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Three-dimensional black hole entropy

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Abstract: We discuss in detail the properties of gravity with a negative cosmological constant as viewed in Cherns-Simons theory on a line times a disc. We reanalyze the problem of computing the BTZ entropy, and show how the demand of unitarity and modular invariance of the boundary conformal field theory severely constrain proposals in this framework.

Keywords: Models of Quantum Gravity, Black Holes.

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1. Introduction

The Einstein-Hilbert action in three dimensions, with a negative cosmological constant (see e.g. [1]) can be rewritten in terms of Chern-Simons theory with gauge group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ [2, 3]. It is worth studying that Chern-Simons theory in detail for several reasons. One reason is that gravitational quantum theories on spaces with negative cosmological constant have been shown to be holographically dual to gauge theories on the boundary, in the context of string theory. The prime example is the correspondence between Type IIB string theory on $AdS_5 \times S^5$ which is dual to $N = 4$ super Yang-Mills [4]. On the other hand we know that Chern-Simons theory on $R \times D$ where D is a disk is dual to a chiral Wess-Zumino-Witten model on the boundary [5, 6]. It should be instructive to study holographic duality in an example that incorporates features of both theories.

Our focus in this paper will be slightly different though. We will first of all map out in detail the correpondence between Chern-Simons states and their spacetime interpretation. We will try to use that knowledge to recompute the entropy of the BTZ black hole [7, 8]. We attempt in two different ways to compute the entropy and show that in the non-unitary theory, there seem to be too many states present to agree with the entropy formula. On the other hand, the minimal unitary truncation has too few degrees of freedom. We thus show how unitarity and the demand of modular invariance of the BCFT seem to overconstrain the black hole counting problem in this context. Our computation differs in crucial respects from those in the literature: we analyze very concretely the boundary conformal field theory, with modular invariant spectrum, that is induced by the Chern-Simons theory on the boundary and pay attention to the modular invariance of the proposed spectrum, as well as to the minimal conformal weight appearing in the theory.

In section 2 we review the Chern-Simons formulation of gravity in three dimensions. In the next section, we discuss quantization of $SL(2,\mathbb{R})$ Chern-Simons theory on a disk with a puncture and map out the correspondence between classical weights and quantum Hilbert spaces. Using that correspondence we make a first connection with BTZ black holes and particle excitations in section 4, and we discuss a gravitational BPS bound. In sections 5 and 6 we analyze in detail why it is difficult to account for the BTZ entropy in this framework.

2. Chern-Simons gravity

We briefly discuss the Chern-Simons formulation of three-dimensional gravity with a negative cosmological constant. We remind the reader that we can write the gravity action in three dimensions as an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons action:[1]

$$\begin{aligned} S[A^+]_{CS} - S[A^-]_{CS} &= \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^+ dA^+ + \frac{2}{3}(A^+)^3\right) - \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^- dA^- + \frac{2}{3}(A^-)^3\right) \\ &= \frac{k}{4\pi l}\int_M \left(e_a \epsilon^{abc} R_{bc} + \frac{1}{3l^2}\epsilon^{abc} e_a e_b e_c\right) + \frac{k}{4\pi l}\int_{\partial M} e^a \omega_a\,, \end{aligned} \tag{2.1}$$

where $A^\pm = \omega \pm \frac{e}{l}$ are two $SL(2,\mathbb{R})$ gauge fields and e is the dreibein and ω is the spin connection one-form. The Newton constant G and the cosmological constant Λ are determined by the formulas $k = \frac{l}{4G}$ and $\Lambda = -l^2$, respectively. Thus the Chern-Simons action is equal to the Einstein-Hilbert action in three dimensions with cosmological constant plus a boundary term, which is half the usual Gibbons-Hawking boundary term (see e.g. [9, 10]). We will henceforth take the Chern-Simons theory as the starting point for studying three-dimensional gravity, without adding any other boundary term. Hence, the consistent boundary conditions on our fields are

$$\int_{\partial M} \mathrm{Tr}(A^\pm \delta A^\pm) = 0\,. \tag{2.2}$$

We should warn at the start that there are fundamental differences between this theory and our intuition for gravitational theories. The most striking difference between the theory of quantum gravity that we obtain in this way and our intuition, is that in the Chern-Simons theory we allow for any gauge connection, including the trivial one which would give rise to a singular geometry in the metric formulation. In other words, the path integral contains an integral over singular geometries. (That is in fact a feature of the quantum theory that makes it renormalizable [3].)

Now that we have established the point of view that we will take on three-dimensional quantum gravity in the Chern-Simons formulation, we make a substantial digression. We first study $SL(2,\mathbb{R})$ Chern-Simons theory in some detail to acquaint the reader with the necessary technical ingredients, without at first instance doubling the degrees of freedom because of the product gauge group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ relevant to the gravitational theory.

[1] Our conventions for $SL(2,\mathbb{R})$ will be: $[T_a, T_b] = \epsilon_{ab}{}^c T_c$ and $\mathrm{Tr}(T_a T_b) = \frac{1}{2}\eta_{ab}$, while we have for the spin connection one-form: $\omega_a = \frac{1}{2}\epsilon_{abc}\omega^{bc}$.

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After this useful digression, we will return to recombine the ingredients for the product gauge group, i.e. the theory of gravitation.

3. SL$(2,\mathbb{R})$ Chern-Simons

Review. We first fix our conventions for Chern-Simons theory. We define the Chern-Simons action as:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right). \tag{3.1}$$

We take the trace in the fundamental (Tr) to have normalization $\mathrm{Tr}(T_aT_b) = \frac{1}{2}\eta_{ab}$ where $\eta_{ab} = \mathrm{diag}(-1,+1,+1)$ and $a \in \{0,1,2\}$. (For compact gauge groups SU(N) (where $\eta_{ab} = \delta_{ab}$), this would lead to the only consistent values for k being integers. For SL$(2,\mathbb{R})$ there is no such restriction at this stage.)

At first, we are interested in Chern-Simons theory on a base manifold M that has the topology of a real line (parametrized by the time t) times a disk with a puncture. To the puncture we associate matter in an irreducible representation of the gauge group. The Chern-Simons action is then supplemented by the particle action $S_p[A,\chi]$:

$$\begin{aligned} S_{\mathrm{coupled}}[A,\chi] &= S_{CS}[A] + S_p[A,\chi] \\ &= S_{CS}[A] + \int dt\, \mathrm{Tr}(\lambda\chi^{-1}(t)(\partial_t + A_t)\chi(t)). \end{aligned} \tag{3.2}$$

For compact gauge groups, λ is a weight for the root system of the Lie algebra of the gauge group. We will see how this statement becomes modified for non-compact groups. We will choose a boundary condition

$$(lA_t \pm A_\phi)|_{\partial M} = 0\,, \tag{3.3}$$

which is consistent with (2.2), and we first perform the path-integration over A_t, which gives the following constraint:

$$\frac{k}{2\pi}F_{12}(t,x^i) + \chi(t)\lambda\chi^{-1}(t)\delta^{(2)}(x^i - P) = 0\,, \tag{3.4}$$

where P denotes the puncture in the disk coordinatized by x^i (where $i \in \{1,2\}$) . We can solve the constraint in terms of a connection that is almost pure gauge, and substitute the solution in the action to obtain the new action

$$\begin{aligned} S[U] = &\frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(U^{-1}\partial_\phi UU^{-1}\left(\partial_t \pm \frac{1}{l}\partial_\phi\right)U\right)dtd\phi + \frac{k}{12\pi}\int_M \mathrm{Tr}(U^{-1}dU)^3 + \\ &+ \frac{1}{2\pi}\int_{\partial M}\mathrm{Tr}\left(\lambda U^{-1}\left(\partial_t \pm \frac{1}{l}\partial_\phi\right)U\right). \end{aligned} \tag{3.5}$$

The system has reduced to a chiral WZW model on the boundary circle coupled to a matter source [6]. For compact groups, we know that when we quantize this system, the Hilbert space is a representation of the chiral current algebra which is the Verma module built on the representation of the gauge group with weight $\lambda - \rho$ (where ρ is half the sum of the positive roots), modded out by the null vectors.

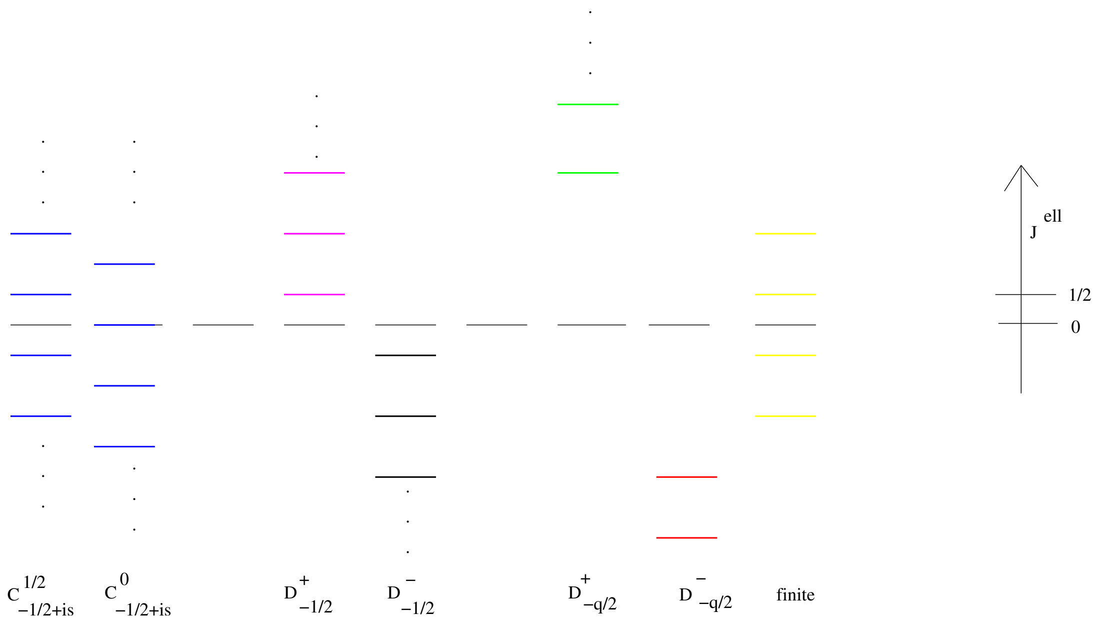

Figure 1: Spectrum of an elliptic operator.

Extension. Now we want to analyze how that last statement is modified in $SL(2, \mathbb{R})$ Chern-Simons theory. To study that problem, it is useful to remind ourselves of how we can quantize a particle on an $SU(2)$ manifold using the method of orbits. Namely, we first concentrate on the particle action in (3.2) and ignore the Chern-Simons action and coupling. We can quantize a particle with resulting quantum spin j, by starting with a classical particle action that is based on the classical weight $(j + \frac{1}{2})\alpha$, where α is the single simple root of $SU(2)$ normalized such that $\alpha^2 = 2$. To show the shift in the weight is non-trivial. The shift in the weight was first obtained in [11], and one could re-obtain it by making use of a proper path-integral quantization, including a non-trivial θ-angle, which would make the computation in [12] more precise. In mathematical terms, the shift is fairly obvious. Indeed, we can build irreducible representations of $SU(2)$ by quantizing appropriate orbits of $SU(2)$ with the canonical symplectic form (which we obtain from the decoupled particle action). We refer to [13] for a pedagogical discussion. In that formalism we find that the quantizable (co)adjoint orbits of $SU(2)$ are the two-spheres with radius $r = n/2$ where n is a strictly positive integer, and that the spin of the representation is indeed related to the sphere of the radius as $j = n/2 - 1/2$, thus accounting for all irreducible representations of $SU(2)$, including the trivial one, using quantizable orbits of maximal dimension two. Luckily this part of the story is readily extendable to $SL(2, \mathbb{R})$. Indeed, we can obtain all irreducible representations of $SL(2, \mathbb{R})$ that occur in the decomposition of the left regular representation by quantizing suitable adjoint orbits. First let us remind the reader of what these irreducible representations are.

All unitary irreducible representations of $SL(2, \mathbb{R})$**.** We give a precise classification of all unitary irreducible representations of $SL(2, \mathbb{R})$ following [14, 15]. The unitary irreps are:

- The continuous representations. There are two series of continuous representations and the series are labeled by the discrete parameter $\epsilon \in \{0, \frac{1}{2}\}$ which indicates the parity of the representation. We denote them by $C^{\epsilon}_{-\frac{1}{2}+is}$ where s is real and the quadratic Casimir c_2 takes the values $c_2 = 1/4 + s^2$ (or $c_2 = -\tau(\tau+1)$ where $\tau = -1/2 + is$ in perhaps more familiar notation). These representations have a spectrum for a normalized elliptic generator $J^{ell} = T^0$ that is either all integers, for $\epsilon = 0$, or all half-integers, for $\epsilon = 1/2$. (See figure 1.)

- The true discrete representations. There are again two series, which are lowest weight representations and highest weight representations. These are denoted D^+_τ or D^-_τ respectively. The index τ takes the values $\tau \in \{-1, -3/2, -2, \dots\}$, and the quadratic Casimir is again $c_2 = -\tau(\tau+1)$. The spectrum for the elliptic generator is given by $-\tau + n$ where n is a positive integer (or zero) for the D^+_τ representation and $\tau - n$ for the D^-_τ representation. (See figure 1.)

- The mock discrete representations. There are two of these, namely $D^{\pm}_{-\frac{1}{2}}$ with spectra similar to the ones described for the true discrete series. (See figure 1.)

- The complementary representations. These have $-1 < \tau < 0$ and $\tau \neq \frac{1}{2}$. They are of even parity.

We make two remarks here. First of all, to the true discrete series with index τ there is naturally associated a finite dimensional non-unitary representation of dimension $-2\tau - 1$ with elliptic eigenvalues that lie between the eigenvalues of the two discrete representations [14]. (See last column in figure 1.) The second remark is that the mock discrete series are the only discrete representations that naturally combine to form a continuous representation as is apparent from the spectrum for the elliptic generator.

Now, the left regular representation on quadratically integrable functions on the group manifold $\mathrm{SL}(2,\mathbb{R})$ decomposes into true discrete representations, and both series of continuous representations (with a known Plancherel measure [15]). These representations only we should be able to obtain by quantizing suitable orbits [13].

The detailed proof of that statement is lengthy. We refer the reader to [13] and just summarize our intuition. First of all, let's draw a picture of the orbits of $\mathrm{SL}(2,\mathbb{R})$ (see figure 2). The structure of the orbits is easily understood when we realize that $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to the Lorentz group in three dimensions $SO(2,1)$. We notice that the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are of several kinds: a paraboloid above the (x^1, x^2) plane, one below the (x^1, x^2) plane, a hyperboloid associated with nearest approach to the x^0-axis, the point at the origin, and a future and past light-cone with the point at the origin removed. Note that the hyperboloid, as we move it towards the origin, when it reaches the origin splits into two light-cone sheets and evolves further into paraboloids.

The correspondence between orbits and representations goes as follows (see also [16]). The hyperboloids of radius s correspond to two different continuous representations $C^{0,1/2}_{-1/2+is}$ — we have two ways of quantizing each orbit. When we pass the hyperboloid through the origin, it splits into two paraboloids. Thereafter, the quantizable orbits are

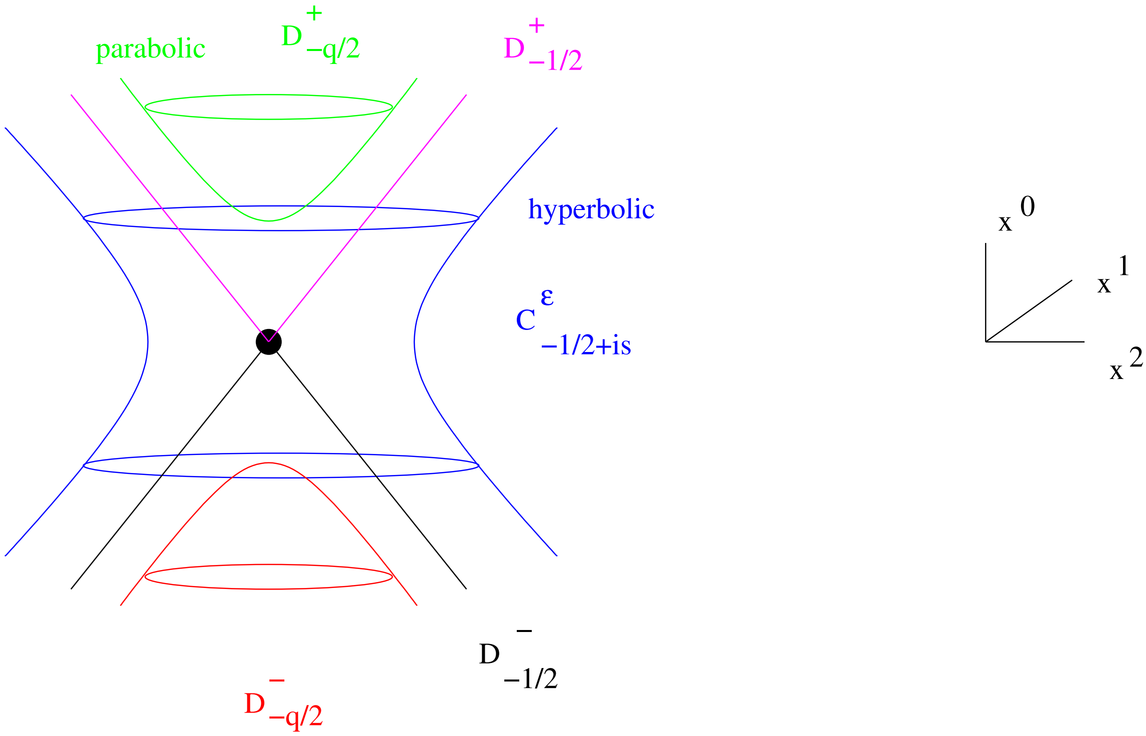

Figure 2: Conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$.

the paraboloids with half-integer radius r, which correspond to the discrete representations $D^{+}_{\tau=-r-1/2}$ and $D^{-}_{\tau=-r-1/2}$ depending on whether we take the upper or the lower sheet.[2] That completes the map of regular representations and quantizable orbits. (Note that the method of orbits gives good intuition for the spectrum of the elliptic operator in a representation associated to a given geometrical orbit. To convince oneself of this fact, the reader should merely try to match up figure 2 with figure 1, which is a useful mental exercise.)

But, from the geometrical picture of orbits, another representation theoretic fact becomes intuitive. The orbits corresponding to continuous representations split up into two light-cones plus the point at the origin when we let s approach 0. We thus gain intuition for the fact that the continuous representation with $s = 0$ (and $\epsilon = 1/2$) decomposes as $D^{\pm}_{-1/2}$, i.e. into two mock discrete representations.[3] Strictly speaking though, the mock discrete representations are not obtained by the orbit method, nor are the complementary representations.

[2]We can think of the doublesheeted hyperboloid as splitting not only into two paraboloids, but also as obtaining radii which differ by a half-integer, since the spectrum of the discrete representations is also either half-integer or integer depending on the value of the radius.

[3]Moreover the gap that arises when we continue further, between the tips of the parabolas corresponding to the discrete representations $D^{\pm}_{\tau}$ is a measure for the size of the non-unitary finite dimensional representations that is naturally associated to these discrete representations. We can picture a ball between the tip of the parabolas corresponding to the well-known spherical orbits of SU(2) and their corresponding finite dimensional representations, after analytic continuation — if we really insist on a pictorial representation.

Conclusion. We take away from these intuitive mathematical facts the following statements about the quantization of a particle with an action determined by a weight in these orbits. When the particle has hyperbolic weight, the associated representation is continuous. When it has non-zero weight on the light-cone, the only invariant under conjugation is the overall sign of the weight. The sign determines whether the associated unitary Hilbert space is $D^+_{-1/2}$ or $D^-_{-1/2}$. When the weight is zero, the representation is trivial. When the weight is elliptic, it is quantized, and the particle Hilbert space will be a discrete representation. (See table 1.) It would be useful to back up these statements with a path integral computation for a particle on an $\mathrm{SL}(2,\mathbb{R})$ manifold, but we refrain from carrying out this exercise in this paper.

The complementary representations are not obtained by the orbit method. They lie in the far quantum regime of the particle action, since the radius of the orbit effectively acts like the particle action coupling constant [16]. The complementary representations clearly are associated to small Casimirs. So, although we have no classical action starting point to study particles in complementary representations, since they are unitary we can accept them as quantum mechanical representation spaces.[4]

In summary, by reviewing the orbit method for obtaining irreducible representations of $\mathrm{SL}(2,\mathbb{R})$, we have hopefully convinced the reader of the fact that we are able to quantize the $\mathrm{SL}(2,\mathbb{R})$ particle action. We have laid out the detailed map between classical weights in the action (3.2) and quantum particle Hilbert spaces (see table 1).

Unitarity and current algebra. When we return to quantizing the full action, including the Chern-Simons term that gives rise to a chiral WZW model on the boundary, we run into subtleties. A naive quantization of the model will give rise to the usual representation of the current algebra on the irreducible representation of $\mathrm{SL}(2,\mathbb{R})$. The model will be non-unitary, because of negative norm states arising from lowering operators associated to the elliptic one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$ (i.e. "the raising operators associated to the time-like direction").

So, we can quantize a particle on $\mathrm{SL}(2,\mathbb{R})$ by analogy to a particle on $\mathrm{SU}(2)$ consistently. But, when we continue the analogy naively to include the action of the current algebra, we arrive at a non-unitary theory. We note at this stage that at least for a bulk without a boundary, there exists a quantization of Chern-Simons theory with non-compact gauge group which is unitary [17]. It would be interesting to figure out whether there is a close analogue of the quantization for the theory on the disk, and if possible, to understand the relevant unitary representation of the current algebra. We will propose to solve the problem of unitarity in two different ways later on.

In passing we note that when we quantize strings on $AdS_3 \times S^3 \times T^4$, for example, then the non-unitarity of the representations of the $\mathrm{SL}(2,\mathbb{R})$ loop group is not a problem after determining the right spectrum, since the ghosts decouple after applying the appropriate constraints (or after computing the BRST cohomology) [18].

[4] In an intuitive sense, these representations exactly appear to fill the hole left by the quantum shift in the classical weight.

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λ	$\pm(-q+1)T_0$ and $q \geq 2$ integer	$2sT_2$	$\pm(T_0+T_2)$	0	–
irrep	$D^{\pm}_{-\frac{q}{2}}$	$C^{0,\frac{1}{2}}_{-\frac{1}{2}+is}$	$D^{\pm}_{-\frac{1}{2}}$	trivial	complementary

Table 1: The correspondence between classical (generalized) weights in the particle action and irreducible quantum Hilbert spaces.

Finally we make a remark on winding sectors. We have not been careful in treating the non-trivial topology of the gauge group with first homotopy group $\Pi_1(\mathrm{SL}(2,\mathbb{R})) = Z$, since it was not crucial for our purposes until now. Later on, we will need the fact that the representation of the current algebra built on the discrete representations satisfies the following relation: $\hat{D}^{+,w}_{-q/2} \equiv \hat{D}^{-,w-1}_{q/2-k/2}$ which was proved in [18]. We could derive it in the Chern-Simons theory by studying the transformation of the Hilbert space when we move from a sector with winding number w to a sector with winding number $w-1$.

4. Black holes and particle excitations

We have reminded ourselves of useful properties of $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory, and we will now bring them to bare on understanding three-dimensional gravity with a negative cosmological constant, with signature $(-1,+1,+1)$, which can be rewritten as a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory as reviewed in section 2. In particular, we will be able to associate holonomies to classical gravity solutions, and we can thus associate them to punctures with particular $\mathrm{SL}(2,\mathbb{R})$ representations in the Chern-Simons theory. We also clarify the role of the massless BTZ black hole solution, of the AdS_3 solution, and of the trivial Chern-Simons background.

Let's study some classical solutions to the equations of motion with a source term. We restrict to the BTZ black hole solutions and compute the corresponding dreibein and spin connection, next to compute the holonomy of the classical gauge field around the source. That provides us with a map between classical sources and holonomies. Next, we can look at the quantum theory with the knowledge acquired in section 3.

To make contact with the usual metric formulation of three-dimensional gravity, we start out with the familiar BTZ metric [7, 8]:

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 r^2 \left(d\phi - \frac{J}{2r^2}dt\right)^2 , \qquad (4.1)$$

where $\phi \in [0, 2\pi[$. The mass M and angular momentum J of the black hole are expressed in terms of the outer and inner horizon $r_\pm$ as $M = \frac{r_+^2 + r_-^2}{l^2}$ and $J = \pm\frac{2r_+r_-}{l}$. The inner and outer horizon are then located at the (positive) square root of:

$$r_\pm^2 = \frac{Ml^2}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right). \qquad (4.2)$$

We choose the following dreibein:

$$
\begin{aligned}
e^0 &= -f dt \\
e^1 &= f^{-1} dr \\
e^2 &= r d\phi - \frac{J}{2r} dt \,,
\end{aligned}
\tag{4.3}
$$

where $f^2 = r^2/l^2 - M + J^2/(4r^2)$. Labeling (t, r, ϕ) tangent directions as $(0, 1, 2)$ respectively, we obtain the one-form:

$$
\begin{aligned}
\omega_{01} &= \frac{r}{l^2} dt - \frac{J}{2r} d\phi \\
\omega_{02} &= -\frac{J}{2r^2} f^{-1} dr \\
\omega_{12} &= -f d\phi \,.
\end{aligned}
\tag{4.4}
$$

From these we easily get the gauge potentials $A^{\pm}$, and see that those potentials satisfy our boundary conditions (3.3) at $r = \infty$: $lA_t^{\pm} \pm A_{\phi}^{\pm} = 0$ holds for arbitrary r. We can compute the gauge invariant expectation value of a Wilson loop looping the origin, and we find [19, 20]:

$$
\mathrm{Tr}_F \, e^{\oint A^{\pm}} = 2 \cosh \left(\pi \sqrt{M \pm \frac{J}{l}} \right) . \tag{4.5}
$$

Now we interpret the formula. First of all, it is clear that the non-trivial black hole solutions correspond to non-trivial source terms for the equations of motion. The holonomy is non-trivial when the field strength is sourced inside the (otherwise) topologically trivial Wilson loop. Thus, we can think of a spatial section of the BTZ space-time as a disk with a puncture where a source is inserted. That establishes the topology of a BTZ space-time.

Next, we notice that a generic BTZ black hole is associated to holonomies that are greater than two, i.e. they correspond to hyperbolic orbits. These orbits are naturally associated to continuous representations of both the left and right $\mathrm{SL}(2, \mathbb{R})$ symmetry group. By comparing the holonomy that we calculated this way to the source term in the coupled Chern-Simons and particle action, we can associate the weight $\frac{2\pi}{k} \lambda^{\pm} = 2\pi \sqrt{M \pm J/l} T_2$ or $\lambda^{\pm} = k \sqrt{M \pm J/l} T_2$ to a black hole with mass M and angular momentum J. It is thus associated to a continuous representation with values $s^{\pm} = \frac{k}{2} \sqrt{M \pm J/l}$.

Next, we analytically continue the formula for the holonomies to negative $M \pm J$ and find:

$$
\mathrm{Tr}_F \, e^{\oint A^{\pm}} = 2 \cos \left(\pi \sqrt{-M \mp \frac{J}{l}} \right) . \tag{4.6}
$$

Following the same reasoning as before, we find the relevant weight for the quantum Hilbert space: $\lambda^{\pm} = \mp k \sqrt{-M \mp J/l} T_0$. We know that when we study the true $\mathrm{SL}(2, \mathbb{R})$ group (and not its covering), the value of λ is quantized as $\lambda^{\pm} = \pm(-q+1) T_0 = \mp k \sqrt{-M \pm J/l} T_0$ $(q \geq 2)$. We thus find that we can only quantize space-times with $k \sqrt{-M \pm J/l} = q - 1$ in that

Chern-Simons theory. We notice for instance that the AdS_3 geometry (— not the covering —) with $J = 0$ and $M = -1$ is only quantizable when k is an integer. This is one way to realize that integer k do play a special role in the $\mathrm{SL}(2,\mathbb{R})$ theory.[5]

Now we address a more subtle point. We notice that a light-cone value for the holonomy, after taking the trace, always gives rise to the trivial value (i.e. two) for the Wilson line. Nevertheless, we know that we can have non-trivial source terms in the classical equation of motion which correspond to light-cone weights. And we know that this statement is gauge invariant (since the light-cones are invariant under Lorentz transformations). Thus, for $M = 0 = J$, we have a special situation, with trivial value for the Wilson loop, but with three different classes of gauge fields which are gauge inequivalent. One class is the trivial gauge connection, associated to the trivial representation. This corresponds to the unbroken vacuum of the Chern-Simons theory. The massless BTZ black hole is non-trivial, and can be associated to the $D^{+}_{-1/2}$ representation, while the massless BTZ black hole with negative values for the radial variable r can be associated to $D^{-}_{-1/2}$. We note that this is not unexpected in view of the fact that on the Poincare patch (AdS) time flows in opposite directions when we continue through $r = 0$, and this is reflected in the fact that the spectrum for the elliptic operator flips sign. (Of course, a lot of these properties are foreshadowed in the non-trivial geometry of the three-dimensional black hole [8].)

An extremal black hole corresponds to a Wilson loop that is trivial (in the sense that it is equal to the Wilson loop of a gauge field that is zero) in either the left or the right sector. It is then associated to a product of a continuous representation and a trivial, or a $D^{\pm}_{-1/2}$ representation, depending on the precise choice of the sign of the associated light-cone weight.

We might also want to study complementary representations of $\mathrm{SL}(2,\mathbb{R})$, i.e. the only remaining unitary representations. These representations can be suitably combined, leading presumably to particle like excitations in AdS_3 (with non-trivial mass and spin). It would be interesting to further study the possible geometric interpretations of more general combinations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ representations. We only note at this point that if we combine complementary or discrete with continuous representations, we violate the bound $|M| \geq |J|$.

Gravitational BPS bound. Let us digress at this point and discuss the gravitational BPS-like bound. We can think of the bound $|M| \geq |J|$ as a BPS bound arising from a supersymmetric version of the Chern-Simons theory that we are studying (see e.g. [21]). Note that we actually should be able to study the quantized theory in this context, and to derive the bound from the action of quantum supercharges on a Hilbert space, in analogy to the derivation in supersymmetric field theory. Imagine we can derive this bound in the quantum theory (for instance by studying the representation theory of the boundary superconformal algebra). Then, it turns out that the quantum gravitational BPS bound is

[5] We note in passing that other even more indirect ways to realize the special role of integer levels is by noticing the spectral flow relation, by identifying the level of the $\mathrm{SL}(2,\mathbb{R})$ theory with quantized charges in string theory, or by trying to combine an $\mathrm{SL}(2,\mathbb{R})$ WZW model with a compact WZW model to obtain a supersymmetric theory.

classically interpreted for the BTZ black holes (with $M \geq 0$) as the condition for absence of a naked singularity, while for $M < 0$, it is interpreted as implying the absence of closed time-like curves. Thus, in the classical bosonic theory the BPS bound arises from physical considerations, while in the supersymmetric quantum theory, we imagine it arising through representation theory. Working out precisely these conceptual connections in the quantum theory will be worthwhile.

5. BTZ entropy I

In this section, we refine the picture described above, and we try to compute the BTZ entropy in the non-unitary theory of quantum gravity described in the previous sections. After analyzing in detail why the computation fails, we formulate a new proposal for a unitary boundary conformal field theory in the next section. However, we will find that unitarity seems to overconstrain the black hole counting problem.

We attempt to compute the BTZ entropy as follows. First, we need to identify which states in Chern-Simons theory on a disk with a source correspond to a BTZ black hole. The generator of time translations $L_0 + \bar{L}_0$ in the boundary theory is the hamiltonian, and determines the mass of the full system (including boundary excitations), which we will identify with the mass of the space-time. Similarly, $L_0 - \bar{L}_0$ corresponds to the total angular momentum. Properly normalizing the Sugawara energy momentum yields (in the semi-classical limit):

$$\begin{aligned} L_0 &= \frac{1}{k-2} : J^a_{-n} J_{a,n} := \frac{1}{16G}(lM + J) \\ \bar{L}_0 &= \frac{1}{k-2} : \bar{J}^a_{-n} \bar{J}_{a,n} := \frac{1}{16G}(lM - J)\,. \end{aligned} \tag{5.1}$$

It is now easy to correct our statements in the previous sections. In section 4 we associated a particular mass and angular momentum to a primary state of the boundary conformal field theory. I.e. the boundary conformal field theory had no oscillator excitations turned on. When we turn on these excitations, we obtain states with a higher total mass. The total mass of the system should be identified as the mass of the space-time. (And analogously for the angular momentum.) Thus, there are many states, built on different ground states, which have the same total value for the mass. The idea behind the black hole entropy counting argument is that we count the number of states corresponding to a given total mass and angular momentum (and fixed topology). We will try to count these states in the semi-classical limit, where the Bekenstein-Hawking entropy formula is valid, i.e. in the limit $l \gg G$ or $k \gg 1$. The formulas in this section should be understood to be valid in this limit only.

To be able to compute that number of states easily, we need a couple of non-trivial ingredients: the bare central charge, the minimal conformal dimension in the theory, and a modular invariant spectrum. The bare central charge of the theory is easy to determine: it is $c = \frac{3k}{k-2}$. (For the connection to the standard result of [22] see [23].) The minimal conformal dimension in the theory depends on the spectrum of the theory, which we want

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to choose in a way consistent with modular invariance. A reasonable proposal for the spectrum seems to be to take the following current algebra representations in the Hilbert space [18]:

$$\hat{D}^{+,w}_{-\frac{q}{2}} \otimes \hat{D}^{+,w}_{-\frac{q}{2}}$$
$$\hat{C}^{\alpha,w}_{-1/2+is} \otimes \hat{C}^{\alpha,w}_{-1/2+is}\,, \tag{5.2}$$

where $-(k-1)/2 < -q/2 < -1/2$ and w is a winding number that labels different sectors of the theory. We also need to specify a measure for the continuous representations which will not be crucial for our purposes. We will discuss subtleties associated to this choice of spectrum a little later.

Given the spectrum, we can determine the minimal conformal dimension and compute the effective central charge of the theory. In the zero winding sector, the minimal conformal dimension is (in the semi-classical limit) $\Delta_{\rm min}^{w=0} = -k/4$, which would give an attractive central charge of $c_{\rm eff} = c - 24\Delta_{\rm min} = 6k = \frac{3l}{2G}$. In turn, that would lead, following the elegant argument of [24] to a swift derivation of the desired BTZ black hole entropy. Unfortunately, this story is not convincing for the following reasons.

Critique. When we analyze the minimal conformal weight in the winding sectors, we find that the conformal weights are not bounded from below. That undermines fatally our first attempt. Note that there would be other valid critiques of the derivation.

One issue that we need to address is the precise modular invariant partition function of the $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten theory. In fact, in the literature we find strong arguments for the proposed spectrum [18] (see also the important footnotes 5, 6 and 18 in [18]), we find the computation of the free energy of string theory on $AdS_3 \times N$, which yields a modular invariant [25] result, and there is moreover an analysis of the factorization of four-point functions in the relevant analytically continued conformal field theory [26]. But it seems that the modular invariant partition function has not been written down as a simple and clearcut formula (see also [25, footnote 2]). What does seem clear is that the winding sectors are crucial to obtain a modular invariant spectrum. That seems sufficiently devastating.

The second most striking shortcoming of our analysis (apart from the bottomless spectrum) is that the boundary theory is non-unitary because states with negative norm can be created by applying appropriate creation operators to primary states. (The representations for the zero-modes are unitary.) To mend this shortcoming, one could try to reproduce the argument in the unitary $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ conformal field theory, and the associated Chern-Simons theory (see e.g. [27]), in other words, in a euclidean setting. The fact that the four-point functions of the euclidean conformal field theory [28] factorize over, amongst others, short string states (i.e. states in discrete representations) then could become an important part of completing the argument.

6. BTZ entropy II

We obtained a non-unitary boundary theory, where modular invariance of the boundary partition function forced a fatal result for the minimal conformal weight. In this section we propose a modified theory which is unitary, and has a clearcut modular invariant partition function with minimal conformal weight. We will show in detail why this modified theory also does not give rise to the right counting of degrees of freedom.

The argument runs as follows. To obtain a unitary theory, we have to get rid off the modes that give rise to negative norm states (at least). The minimal way to achieve this, for Chern-Simons theory on a disk, is to add a boundary coupling. We choose the boundary coupling to be an interaction term that includes a boundary gauge field that gauges the elliptic U(1) subgroup of $\mathrm{SL}(2,\mathbb{R})$ at the boundary. We thus leave the bulk gravitational theory intact, and modify the action by a boundary term only. The resulting boundary model will be the coset two-dimensional conformal field theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. The resulting gravitational theory has a space-time hamiltonian which is the coset conformal field theory hamiltonian (since that is the operator that generates time translations), and a similar reasoning holds once again for the angular momentum. We obtain:

$$\begin{aligned} L_0^{cs} &= L_0^{\mathrm{SL}(2,\mathbb{R})} - L_0^{U(1)} = \frac{k}{4}\left(M + \frac{J}{l}\right) \\ \bar{L}_0^{cs} &= \bar{L}_0^{\mathrm{SL}(2,\mathbb{R})} - \bar{L}_0^{U(1)} = \frac{k}{4}\left(M - \frac{J}{l}\right), \end{aligned}$$

where the allowed states are the ones with $J^{\pm}_{-n}$ excitations and with $J_0^0 - \bar{J}_0^0 = n$ and $J_0^0 + \bar{J}_0^0 = -kw$. The numbers n and w are U(1) charges under the unbroken U(1) subgroups of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, and in the context of the cigar CFT they are associated to non-trivial winding and momentum in the angular direction [29].

Let us discuss a few aspects of our proposal for a modified, unitary theory. The truncation of the boundary theory might seem arbitrary. We argue that it is the minimal truncation that preserves unitarity. The original theory in fact contains ghosts and can intuitively be argued to have too many states to obtain the correct entropy law. The truncation we proposed above is the one that precisely only gets rid of the states responsible for non-unitarity. At the same time, we note that the requirement of a precise modular invariant partition function and spectrum also constitute an argument for the proposed truncation. If we desire the boundary conformal field theory to be consistent, a spectrum giving rise to modular invariance seems imperative.

The truncation moreover seems to interfere with perhaps desirable gluing properties of Chern-Simons theory. It may be thus useful to remark that our truncation may be achieved in another manner. We could include in our action a bulk Chern-Simons $\mathrm{U}(1)\times\mathrm{U}(1)$ action, and include a ghost boundary term, and moreover demand that our boundary states are in a given BRST cohomology (see [29]). That will lead, in the particular example of the theory on the disk to the same boundary action and boundary theory. The decoupled bulk equations still allow for the standard gravity solutions.

But, in any case, the truncation gives rise to a special role for the $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry that we gauged. In fact, we are left with only a $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry (which conformal

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field theorists may think off as associated to the angular direction of the semi-infinite cigar or two-dimensional black hole) which in our context is the charge of two components of the dreibein or of the spin connection under a rigid rotation. The assignment of these charges and their incorporation in the spectrum are crucial for modular invariance. The fact that our fundamental variables are the dreibein and the spin connection is thus important. (In this respect, our formalism is reminiscent of loop quantum gravity.)

We note also that the demand of modular invariance incorporates other non-trivial information (compare e.g. to [30]) as seen below. It implies that the two chiral sectors of the theory are linked. Although the two sectors are already related by the demand of quantization of space-time angular momentum, our insistence on a modular invariant spectrum also locks the zero-modes of the two chiral sectors, and chooses to combine the left and right spectrum in a very particular way.

Now let us compute the black hole entropy in this unitary theory. In this theory the argument for black hole state counting can be made precise, but it does not lead to the expected result. The bare central charge is $c = \frac{3k}{k-2} - 1$, and the spectrum of states needed for an explicitly modular invariant partition function is given by [31]:

$$\begin{gathered} \hat{D}^{+}_{-\frac{q}{2}} \otimes \hat{D}^{+}_{-\frac{q}{2}} \\ \hat{C}^{\epsilon}_{-1/2+is} \otimes \hat{C}^{\epsilon}_{-1/2+is} \end{gathered} \tag{6.1}$$

where $-(k-1)/2 < -q/2 < -1/2$ and J_0^0 and $\bar{J}_0^0$ satisfy the constraints $J_0^0 - \bar{J}_0^0 = n$ and $J_0^0 + \bar{J}_0^0 = -kw$. The conformal weights of the primaries are given by:

$$\begin{aligned} h_{\text{primary}} &= -\frac{q(q-2)}{4(k-2)} + \frac{(n-kw)^2}{4k} \\ \bar{h}_{\text{primary}} &= -\frac{q(q-2)}{4(k-2)} + \frac{(n+kw)^2}{4k}\,. \end{aligned} \tag{6.2}$$

Here we see explicitly that the space-time angular momentum is quantized. To determine the minimal conformal weight, we need to take into account the spectrum of the elliptic generator in the discrete lowest weight representations (as discussed in section 3), and we find that the U(1) contribution to the conformal weight (i.e. the second term in formula (6.2)) forces a minimal conformal weight that is positive, consistent with the unitarity of the cigar conformal field theory.[6] The effective central charge is then too small to be able to account for the expected black hole entropy.

In summary, we analyzed in detail two proposals for boundary conformal field theories, and showed that both in the non-unitary theory and in the unitary theory, the demand of modular invariance for the boundary partition function forced a fatal result for the effective central charge. It seems that even a minimal unitary truncation of the unitary theory does not allow for a sufficient number of degrees of freedom to account for the black hole entropy. Indeed, in every unitary framework, the effective central charge will not be larger then the

[6]We thank Juan Maldacena for pointing out a crucial oversight on this point in the original version of our paper which lead to a faulty conclusion.

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bare central charge. Although one would like to appeal to a negative conformal weight for an AdS_3 ground state to raise the effective central charge, that is difficult to reconcile with unitarity.

Some comments on our analysis are in order. Our analysis should be compared to those in the literature. (see e.g. [24] and [32]–[38]). For a critique of the approaches in [32, 33, 24] we refer to the paper [39]. The main difference between these papers and ours is that we explicitly identify proposals for the dual conformal field theory, and the degrees of freedom that should be responsible for the black hole entropy.

There has been some discussion in the literature of what the relevant degrees of freedom are, and where they are located. In our theory they are clearly the degrees of freedom located at the boundary of the manifold (and associated to the punctures on the disk). The boundary can be viewed as asymptotic infinity, as it would in an approach as in [24], or as located on the horizon, as in [40]. From our perspective, we need to identify the full system, both the punctured disk and the boundary degrees of freedom as describing a specific black hole state. The total mass of the system, which we identify with the black hole mass, is given by both contributions from the puncture (the weight of the primary) and the excitations at the boundary. From this viewpoint, it may perhaps be more natural to think of our system as describing the interior of a black hole, say the space beyond the event horizon. If these statements seem counterintuitive, we remark that all BTZ black hole entropy counting arguments in the literature share this property with our proposal.

We further remark that also in the framework of the metric formulation of three-dimensional gravity (with AdS_3 boundary conditions [22]), it may be difficult to account for the BTZ entropy. The connection between the gravity formulation and our unitary treatment can be established by studying the supergravity theory, with an ($N = 2$) super-Liouville theory on the boundary [41, 42, 30], which is conjectured[43, 44] / proven [45, 46] to be dual to the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ supercoset model.[7] Since the supercoset is expected to share the crucial features of the theory we studied, it may be equally difficult to compute the (standard) black hole entropy formula within this framework, since the effective central charge of the coset theory (or the Liouville theory) will not be sufficiently high.

7. Conclusions

We conclude our long discussion with a brief summary. In this paper, we have analyzed the Chern-Simons formulation of three-dimensional gravity. In particular, we have mapped out the Hilbert spaces for $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory associated with particular classical sources. We thereby clarified some of the vacuum structure of three-dimensional gravity with a negative cosmological constant, and we made a proposal for a concrete investigation of a quantum gravitational BPS bound. Moreover, we have analyzed in detail the boundary conformal field theories that arise in the non-unitary formulation, and in a minimal truncation that is unitary. We showed that the demand of a modular invariant partition function, and unitarity leaves little room for making the BTZ black hole counting argu-

[7] See also [47].

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ment precise in this context. It will be interesting to understand whether some consistency requirements can be relaxed or whether different topologies can be summed over, in order to find an explicit model for black hole entropy counting in this simple framework.

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